Journal of Sound and Vibration (2001) **239**(1), 57–67
doi:10.1006/jsvi.2000.3156, available online at http://www.idealibrary.com on IDEAL®

VIBRATION OF A BEAM WITH A BREATHING CRACK

T. G. CHONDROS

Department of Mechanical Engineering and Aeronautics, University of Patras GR 26110 *PATRAS, Greece. E-mail: chondros@mech.upatras.gr*

A. D. DIMAROGONAS

Washington University, St. Louis, Missouri, U.S.A

AND

J. YAO

Siemens Medical Laboratories, Concord, CA, U.S.A.

(*Received* 2 *March* 1999, *and in final form* 23 *May* 2000)

A continuous cracked beam vibration theory is used for the prediction of changes in transverse vibration of a simply supported beam with a breathing crack. The equation of motion and the boundary conditions of the cracked beam considered as a one-dimensional continuum were used. The eigenfrequency changes due to a breathing edge-crack are shown to depend on the bi-linear character of the system. The associated linear problems are solved over their respective domains of definition and the solutions are matched through the initial conditions. The changes in vibration frequencies for a fatigue-breathing crack are smaller than the ones caused by open cracks. The method has been tested for the evaluation of the lowest natural frequency of lateral vibration for beams with a single-edge breathing crack. Experimental results from aluminium beams with fatigue cracks are used for comparison with the analytical results.

1. STATE OF THE ART

The presence of a crack in a structural member introduces a local flexibility that affects its dynamic response. Moreover, the crack will open and close in time depending on the loading conditions and vibration amplitude. The changes in dynamic characteristics can be measured and lead to an identification of the structural changes, which eventually might alead to the detection of a structural flaw. A wealth of analytical, numerical and experimental investigations now exists [1]. The local flexibility of a crack has been widely used in the past 15 years for vibration analysis of cracked beams. This spring hinge model was combined with the fracture mechanics results for crack identification in various structures [2]. The theoretical shortcomings of the linear spring model for the crack in high frequencies put a limitation on the development of the dynamic response theory of the cracked structural elements. On the other hand, most of the researchers assumed in their work that the crack in a structural element is open and remains open during vibration. Such as assumption was made to avoid the complexities that resulted from the non-linear characteristics presented by introducing a breathing crack.

During the vibration period of a cracked structural member, the crack does not remain always open. The static deflection due to some loading component on the cracked beam (residual loads, body weight of a structure, etc.) combined with the vibration effect may cause the crack to open at all times, or open and close regularly, or completely close depending on various loads at a given time. If the static deflection due to some loading component on the beam (dead loads, own weight, etc.) is larger than the vibration amplitudes, then the crack remains open all the time, or opens and closes regularly and the problem is linear. If the static deflection is small, then the crack will open and close in time depending on the vibration amplitude. In this case the system is non-linear.

Due to the lack of a systematic theory regarding the breathing crack, it is difficult to interpret the experimental results. The effect of the breathing crack in the vibration response of cracked structural members had been recognized long ago. Kirmsher in 1944 [3] reported that if a crack in a concrete beam is filled with dirt or crystallized material, or is narrow enough so that interference occurs, the effect on the natural frequency is the same as that of a crack of lesser depth. This observation was the basis for a more systematic investigation of the effects of opening and closing of cracks.

Actis and Dimarogonas [4], used the finite element method to study the simply supported cracked beam. The crack was assumed to be a breathing crack. They assumed that when the bending moment changes sign, the crack changes from open to closed, or from closed to open. When the crack is open, an additional stiffness $-[\delta K]$ would be introduced. Thus, the linear algebraic equation of the uncracked structural member

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\}$$

would change to

$$[M]\{\ddot{u}\} + [K - \delta K]\{u\} = \{F\},$$

where $[M]$ is the mass matrix, $[K]$ is the stiffness matrix, $\{u\}$ is the displacement vector and $\{F\}$ is the load vector.

The words "spring hinge model" used in the literature refer to the linear spring model, because, for a given crack depth, the equivalent spring constant remains the same for both directions of loading. In the case of the breathing crack, a crack that opens and closes during vibration, the spring constant appears to be different for open and closed cracks. The crack behaves as a bi-linear spring. Additionally, the dynamic response of the bi-modulus material also behaves like a bilinear spring.

Ambartsumyan and Khachatryan [5] developed a different-moduli theory of elasticity. Khachatryan [6] applied it to the longitudinal vibration of prismatic bars made of different-moduli materials. Lenkov and Tolokonnikov [7] studied the axisymmetric strains in materials with different moduli. Green and Mkrtichian [8], Paolinelis *et al.* [9], Bert and Kumar [10], Tran and Bert [11], Reddy [12] and Doong and Chen [13] also referred to a bimodulus material because of its potential application in composite materials.

Ibrahim *et al.* [14] presented a bondgraph technique that models the crack as a torsional spring with two spring constants, one when it is open, and the other when it is closed. A numerical simulation procedure is used for the prediction of the non-linear behaviour of a cantilever beam with a fatigue crack located near its root.

Qian *et al.* [15] investigated the effects of an opening and closing crack on the dynamic behaviour of a cantilever beam using a finite element model for the cracked member. A numerical method and Hermitian interpolation was introduced for the solution of the resulting non-linear equations of motion.

Bayly [16] and Happawana *et al.* [17] studied the effects of the magnitude of perturbations (disorders) on the localization of modal shapes for non-linear vibrating systems. By applying the regular perturbation technique to the characteristic equation of the system, they obtained algebraic expressions for the eigenvalues as a power series in the small parameter or perturbation with acceptable accuracy.

Chu and Shen [18] presented an approximate analytical technique to predict the superharmonic components resulting from low-frequency excitation of an undamped bilinear oscillator. The presence of superharmonic components in the Fourier spectrum has been proposed as an indicator of the discontinuity in stiffness resulting from a breathing crack in the structural element.

Shen and Chu [19] investigated the existence of fatigue cracks by exciting the structures at different frequencies and using a numerical study for the response analysis.

Chondros *et al.* [20] developed a continuous cracked beam vibration theory for the lateral vibration of cracked Euler–Bernoulli beams with single or double-edge cracks. This continuous cracked beam vibration theory is used for the prediction of the dynamic response of a simply supported beam with open surface cracks.

The results of two independent evaluations of the lowest natural frequency of lateral vibrations of a beam with a single-edge surface crack are reported: a numerical solution based on the continuous crack flexibility vibration theory, and an asymptotic solution for the breathing crack developed here. The above analytical results were correlated with experimental results obtained on aluminium beams with open fatigue cracks and breathing cracks respectively. Numerical and experimental results show a substantial variation of the frequency changes for a cracked beam relative to an open-edge crack, with changes computed with the analytical method for a breathing crack.

2. THE EQUATION OF MOTION

A simply supported Euler–Bernoulli beam of length L_0 with an open single transverse surface crack is shown in Figure 1. Let the displacement components be denoted by u_i, the strain components by γ_{ij} and the stress components by σ_{ij} with $i, j = 1,2,3$ referring to Cartesian axes x, y, z. Let p_i be the momentum such that $T_m = 1/2\, \rho\delta_{ij}p_i p_j$ will be the kinetic energy density (δ_{ij} is Kronecker's delta). (A list of nomenclature is given in the Appendix.) For arbitrary independent variations δu_i, $\delta\gamma_{ij}$, $\delta\sigma_{ij}$, and δp_i, the extended Hu–Washizu variational principle, Christides and Barr [21] Chondros *et al.* [20], was introduced in the form

$$\int_V \Big\{ [\sigma_{ij,j} + F_i - \rho\dot{p}_i]\delta u_i + [\sigma_{ij} - W_{,\gamma_{ij}}]\delta\gamma_{ij} + \left[\gamma_{ij} - \left(1 - \frac{1}{2}\delta_{ij}\right)(u_{i,j} + u_{j,i})\right]\delta\sigma_{ij} + [\rho\dot{u}_i - T_{m,p_i}]\delta p_i \Big\} \mathrm{d}V + \int_{S_g} [\bar{g}_i - g_i]\delta u_i \,\mathrm{d}S + \int_{S_u} [u_i - \bar{u}_i]\delta g_i \,\mathrm{d}S = 0, \tag{1}$$

where, $W(\gamma_{ij})$ is the strain energy density function, ρ is the density of the material. F_i, g_i and u_i are, respectively, the body forces, the surface traction and the surface displacement. Moreover, V is the total volume of the solid and S_g and S_u are its external surfaces. The overbar denotes the prescribed values of the surface traction and the surface displacement.

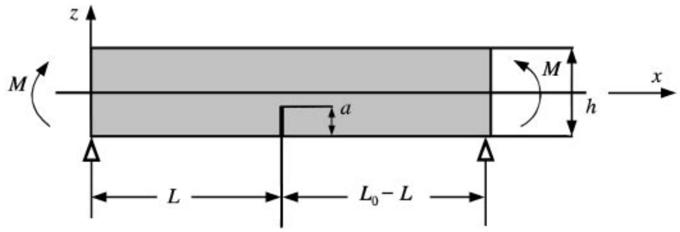

Figure 1. Geometry of a simply supported beam with an edge crack.

The prescribed surface tractions g_i are applied over the surface S_g and the prescribed displacements u_i are over S_u. Together, S_g and S_u make up the total surface of the solid. The differentiation with respect to time ($\partial/\partial t$) is indicated by a dot. Commas in the subscripts indicate differentiation with respect to Cartesian axes.

The change in stress, strain and displacement distributions due to the crack will be expressed by a crack disturbance function for the axial displacement $f(x, z)$ introduced in reference [20].

For a uniform beam in the absence of body forces, the introduction of the displacement disturbance function $f(x, z)$ will lead to

$$u_x = -z\{[1 + f(x, z)]w(x, t)\}', \; u_y = 0, \; u_z = [1 + f(x, z)]w(x, t), \; p_x = 0, \; p_y = 0, \; p_z = P(x, t),$$

$$\gamma_{xx} = -zS(x, t), \; \gamma_{yy} = \gamma_{zz} = -\nu\gamma_{xx}, \quad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0,$$

$$\sigma_{xx} = -zT(x, t), \; \sigma_{xz} = \sigma_{xz}(x, z, t), \; \sigma_{xy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0, \; F_x = F_y = F_z = 0. \quad (2)$$

The term σ_{xz} is introduced for the lateral loading of the beam. Following the method introduced in references [20, 21] equations (2) can now be substituted into the general variational equation (1) and independent variations of the unknowns w, P, S and T can be considered. The above variations considered one by one [20] lead to the general equation of motion

$$E[(I - I_2)(Q_1 w'' + Q_2 w' + Q_3 w)]'' + I_7 \ddot{w} = 0, \quad (3)$$

where E is the Young's modulus of elasticity, $Q_1(x) = I_6/(I - 2I_2)$, $Q_2(x) = 2I_5/(I - 2I_2)$, $Q_3(x) = I_4/(I - 2I_2)$ and the various integrals over the cross-section A are defined as $I = \int_A z^2 \, dA$, $I_2 = \int_A z \, dA$, $I_4 = \int_A z^2 f'' \, dA$, $I_5 = \int_A z^2 f' \, dA$, $I_6 = \int_A z^2 (1 + f) f \, dA$ and the velocity momentum term as $I_7 = \int_A (1 + f) \, dA$.

The displacement disturbance function $f(x, z)$ is calculated in reference [20] as

$$f = -6\pi\,(1 - \nu^2) h^2 \, \Phi_I(\alpha)(x - L)^2 / z L_0 (L_0^2 + \nu h^2/4),$$

where

$$\Phi_I(a) = 0{\cdot}6272a^2 - 1{\cdot}04533a^3 + 4{\cdot}5948a^4 - 9{\cdot}9736a^5 + 20{\cdot}2948a^6 - 33{\cdot}0351a^7$$
$$+ 47{\cdot}1063a^8 - 40{\cdot}7556a^9 + 19{\cdot}6a^{10} \text{ and } \alpha = a/h.$$

Upon denoting by $\bar{u}$, $\bar{X}$ the prescribed displacements and the prescribed external forces respectively, the boundary conditions appropriate to the equation of motion (3) for a simply supported beam [20] are

$$\bar{w} = 0, \; \bar{X} = 0 \quad \text{at } x = 0 \quad \text{and} \quad \bar{w} = 0, \; \bar{X} = 0 \quad \text{at } x = L_0. \quad (4)$$

For a cracked beam with a rectangular cross-section of height h and width b (see Figure 1), the equation of motion (3) for an open crack [20] changes to

$$c_0^2[(I_7 w)^{iv}] + I_7 \ddot{w} = 0, \tag{5}$$

where $c_0^2 = EI/(\rho A)$ is a material constant, and I the appropriate second moment of area.

From equation (5) it can be seen that the displacement disturbance factor $f(x, z)$ affects directly the displacement $w(x, t)$ through the function $I_7(x)$. The appropriate boundary conditions and initial conditions will be used to solve the last differential equation (5).

If cracks are absent from the beam, the functions $f, I_2, I_3, I_4, I_5, Q_2, Q_3$ are zero, Q_1 is unity and the function I_7 is replaced by area A. The equation of motion (3) will reduce to

$$EI\partial^4 w(x, t)/\partial x^4 + A\rho\partial^2 w(x, t)/\partial t^2 = 0. \tag{6}$$

3. NATURAL FREQUENCIES-EDGE CRACK

Let a beam as shown in Figure 1, bent by a pair of symmetrical bending moments M applied at both ends at an instant of time $t_0 = 0$, be suddenly released and perform vibration freely. The boundary conditions obtained from equations (4) are

$$w|_{x=0} = 0, \partial^2 w/\partial x^2|_{x=0} = 0, \quad w|_{x=L_0} = 0, \quad \partial^2 w/\partial x^2|_{x=L_0} = 0. \tag{7}$$

Following the method of separation of variables, the general solution of equation (5) can be written as

$$w(x, t) = W(x)\, T(t), \tag{8}$$

where $T(t)$ is a function of time. By this form of the solution it is assumed that every point of the beam has harmonic vibration of circular frequency ω and amplitude $W(x)$.

Substituting the above solution, equation (8), into the partial differential equation (5), one obtains

$$c_0^2 \left\{ \partial^4 \left[\frac{I_7(x) W(x)}{\partial x^4} \right] \right\} T(t) = [I_7(x) W(x)] \frac{\partial^2 T}{\partial t^2} = 0. \tag{9}$$

This partial differential equation for the flexural vibration of cracked beams can be broken up into two ordinary differential equations,

$$[I_7(x) W(x)]^{iv} + (\omega_n^*/c_0)^2 [I_7(x) W(x)] = 0\,, \quad \ddot{T} + \omega_n^{*2} T = 0, \tag{10, 11}$$

where ω_n^* are the natural frequencies of the cracked beam.

Equation (10) is the differential equation for the natural modes of vibration of the beam considered as a continuous system. The solution of the latter equation was investigated in reference [20] for the case of a cracked beam with an open crack. The solution was found to be

$$W(x) = \phi(x)\, [A_n \sin(\beta_n^* x) + D_n \sinh(\beta_n^* x)], \tag{12}$$

where $\omega_n^* = c_0 \beta_n^{*2}$ are the natural frequencies of the cracked beam, A_n and D_n are constants, and $\phi(x) = 1/I_7(x)$ is the shape disturbance function associated with the crack disturbance function $f(x, z)$ [20].

The boundary conditions at $x = 0$ and L_0, equations (4), yield the characteristic equation

$$I_7'(L_0)/I_7(L_0)\,[\cos(\beta_n^* L_0) - \sin(\beta_n^* L_0)\coth(\beta_n^* L_0)] + \beta_n^* \sin(\beta_n^* L_0) = 0. \quad (13)$$

This implicit natural frequency equation (13) is solved directly for an exact solution β_n^* through a numerical method and the results are shown in Figure 3 of section 4.

4. BREATHING CRACK

For the beam with a breathing crack it is assumed in this investigation that this is a piecewise linear system. This bilinear-type breathing crack has only two states, either fully open or fully closed, as shown in Figure 2, and the frequency does not depend on amplitude. It is also assumed that the transition period from open to closed crack occurs at times when the beam returns to its undeformed shape. Due to the bi-linear character of the system there is no single frequency of oscillation. Instead, there is a dominant frequency of oscillation, which is periodic and depends on the breathing crack behaviour. The solution of the second ordinary differential equation (11) will be

$$T_n = M_n^* \sin(\omega_n^* t) + M_n \cos(\omega_n^* t), \quad (14)$$

where M_n^* and M_n are constants determined by the initial conditions.

By combining the solutions (12) and (14) of the two equations, the general solution of the partial differential equation (5) can be written in the form

$$w(x, t) = \sum_{n=1}^{\infty} \phi(x)[A_n \sin(\beta_n^* x) + D_n \sinh(\beta_n^* x)][M_n^* \sin(\omega_n^* t) + M_n \cos(\omega_n^* t)]. \quad (15)$$

The frequency β_n^* has the form

$$\beta_n^{*4} = \omega_n^{*2}/c_0^2. \quad (16)$$

A_n and D_n can be found from equation (12), and constants M_n^* and M_n can be determined from the initial conditions.

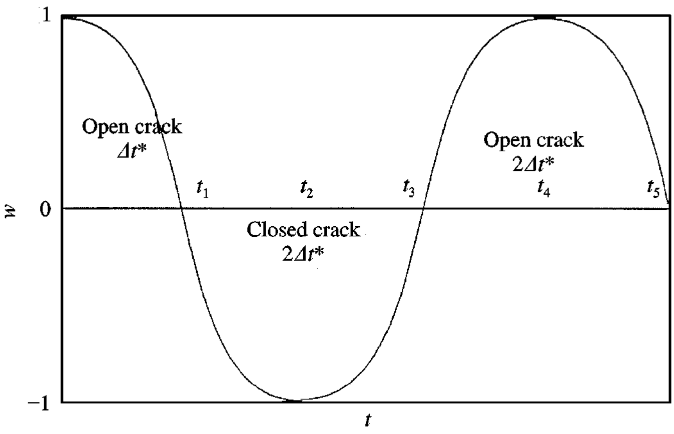

Figure 2. Transverse motion of a simply supported prismatic beam with a single-edge crack at mid-span initially bent to its first mode.

If the crack is not present, equation (15) of the cracked beam will change to

$$w(x, t) = \sum_{n=1}^{\infty} [A_{cn} \sin(\beta_n x) + D_{cn} \sinh(\beta_n x)][M_n^* \sin(\omega_n t) + M_n \cos(\omega_n t)] \tag{17}$$

of the uncracked beam, where the parameters A_{cn}, D_{cn}, M_n^*, and M_n can be computed from the initial conditions.

Since the solution of the bi-linear vibration of the cracked beam depends on the initial conditions, different initial conditions will give different results. One type of initial condition, a cracked beam initially bent to its first mode, will be discussed here.

Consider a beam with a crack initially opened by bending it to its first mode at time $t_0 = 0$ and then release it (see Figure 2). The initial conditions are

$$w|_{t=0} = A_1 \phi(x) \sin(\beta_1^* x), \; \partial w/\partial t|_{t=0} = 0, \tag{18}$$

where A_1 is a known constant. By applying the initial conditions in equation (13) for the beam with an open crack, the constant parameters are determined as

$$A_1 \neq 0, A_n = 0, (n \neq 1), D_n = 0, M_1 = 1, M_n = 0, (n \neq 1), M_n^* = 0. \tag{19}$$

Higher frequencies decay faster than lower frequencies and this is supported by experimental results which show that after a few cycles the fundamental frequency is dominant. Then, for the fundamental mode of vibration, equation (15) is reduced to

$$w(x, t) = A_1 \phi(x) \sin(\beta_1^* x) \cos(\omega_1^* t), \quad (t_0 < t < t_1). \tag{20}$$

At the time t_1 (see Figure 2), the crack is expected to close: that is,

$$w(x, t_1) = 0. \tag{21}$$

Then, equation (20) will yield

$$t_1 = \Delta t^* = \frac{\pi}{2(\pi/L_0 - \varepsilon_1)^2} \sqrt{\frac{\rho A}{EI}}, \tag{22}$$

where $\varepsilon_1 = \beta_1 - \beta_1^*$. At the time instant t_1, the particles of the beam move with velocity $-d_2(x)$ that can be also determined from equation (20) as

$$-d_2(x) = \frac{\omega_1^*}{I_7(x)} \sin(\beta_1^* x) \sin(\omega_1^* t_1). \tag{23}$$

After the time instant t_1, the crack will close and stay closed. The previous initial conditions for the open crack are no longer valid. They will change to

$$w|_{t=t_1} = 0, \quad \left.\frac{\partial w}{\partial t}\right|_{t=t_1} = -d_2(x) \tag{24}$$

of the closing crack (see Figure 2). Also, the solution (15) of the open cracked beam is no longer valid. The solution, equation (17), of the uncracked beam will be used instead.

Substituting the first initial condition $w(x, t_1) = 0$ into equation (15) yields $M_n^* = 0$. Constant M_n can be determined by applying the second initial condition of equations (18) in equation (17). Again, upon considering the dominating fundamental mode of vibration at the time instant $t_2 = t_1 + \Delta t$ (see Figure 2), by setting $\dot{w}|_{t=t_1+\Delta t} = 0$, equation (17) yields

$$\Delta t = \frac{L_0^2}{2\pi}\sqrt{\frac{\rho A}{EI}}. \tag{25}$$

At the time instant $t_2 = t_1 + \Delta t$, the motion of the particles is reversed in the opposite direction. At the time $t_3 = t_1 + 2\Delta t$, the displacement of all points of the beam becomes zero, i.e., $w(x, t_3) = 0$ and the beam returns to its non-deformed state. The velocity of the particles of the beam at the time t_3 is $\dot{w}|(x, t_3) = d_2(x)$. Starting at the time instant t_3 the beam will be bent and the crack will reopen again.

The period of vibration for the breathing cracked beam is

$$T = 2(\Delta t^* + \Delta t). \tag{26}$$

Consequently, the frequency ratio is

$$\omega_b = 2\pi/T = \frac{2\pi}{2(\Delta t^* + \Delta t)} \tag{27}$$

or

$$\frac{\omega_b}{\omega_1} = 2\frac{\omega_1^*/\omega_1}{1 + \omega_1^*/\omega_1}, \tag{28}$$

where ω_b is the frequency of the breathing crack, ω_1 is the frequency of the closed crack and ω_1^* is the frequency of the open crack. Figure 3 shows the lowest natural frequency ratio ω_b/ω_1 for the transverse vibration of a simply supported beam with a surface breathing crack at mid-span, versus the crack depth ratio $\alpha = a/h$.

5. EXPERIMENTAL EVIDENCE

Prismatic beams made of aluminium of rectangular cross-section 7×23 mm and length 235 mm were prepared. Material properties are Young's modulus of elasticity $E = 7{\cdot}2$ E10 N/m^2 and material density 2800 kg/m^3. At mid-span, a sharp notch was introduced perpendicular to the longitudinal axis and the longer dimension of the cross-section. Then, the beam was placed on a shaker table, with one end fixed and the other free and it was vibrated at its lowest bending natural frequency for the purpose of initiating and propagating a fatigue crack. Different specimens were vibrated at different numbers of cycles so that different crack lengths were obtained. Thus, 30 specimens were prepared with crack depths varying from 5 to 60% of the cross-section height.

Then, each beam was simply supported at the two ends by sharp knife-edge steel supports to assure free flexural motion. A small accelerometer of mass 1 g was fixed at mid-span on the surface of the beam opposite to the crack. The vibration frequency was calculated by measuring the time that elapsed for 50 cycles of vibration. Moreover, an FFT transform was performed at the stored signal for an independent measurement of the flexural natural frequencies. The lowest natural frequency of the short aluminium beams was around 2 kHz. The 100 kHz sampling rate two-channel A/D converter used gave good accuracy for the fundamental frequency measurement.

Two sets of experiments were carried out. First, a series of tests were performed on the specimens with a gradually increasing load applied through a spring, opposite to the crack,

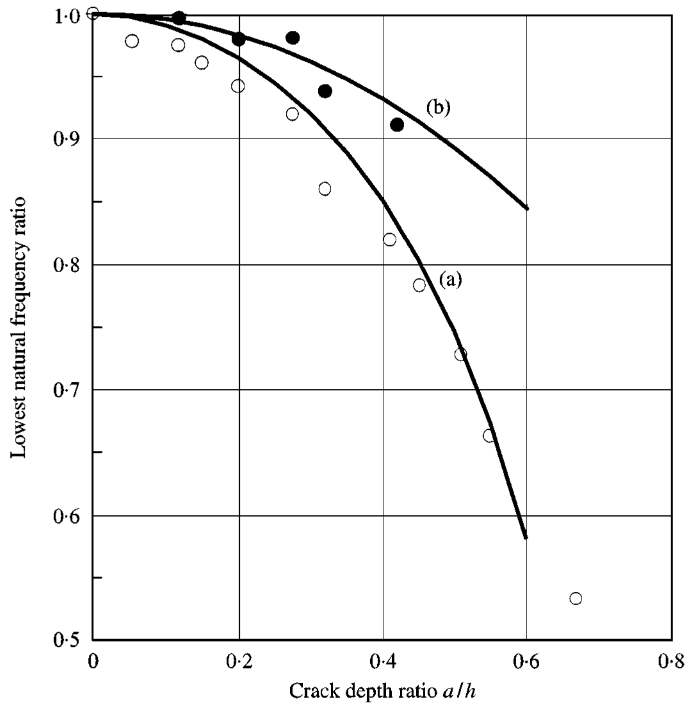

Figure 3. Lowest transverse natural frequency ratio for a simply supported beam with a surface crack at mid-span, versus the crack depth ratio $\alpha = a/h$. Analytical results: (a) continuous crack model, equation (13); (b) breathing crack, equation (28). Experimental results: ○ open crack; ●, breathing crack.

to ensure the transition of the beam dynamic behaviour from that with a closed crack to that with an open crack. Experimental results were in agreement with the breathing crack behaviour, equation (28). Measurements were taken for crack depths up to 40% of the width height, since for larger depths the cracks were partly open due to the crack formation procedure followed. Then, the cracks were forced open and flexural vibration tests were repeated. Experimental results comply with the open crack theory, equation (13). Measurements with open cracks were taken for crack depths up to 60% of the width height, which is of importance for engineering applications. For both sets of measurements the experimental points are averages from tests but the spread of frequency measurements about the points was very small.

6. CONCLUSIONS

Most of the researchers in the literature cited assumed in their work that the crack in a structural element is open and remains open during vibration. This assumption was made to avoid the complexities arising from the non-linear characteristics presented by introducing a breathing crack. With the consistent one-dimensional cracked beam theory used, an analytical approach to the bi-linear dynamic problem of the cracked beam has been developed here. The crack was modelled as a continuous flexibility by using the displacement field in the vicinity of the crack, found with fracture mechanics methods. The effect of the breathing crack on the natural frequencies of vibration of a simply supported beam was studied by solving piecewise linear equations and appropriate continuity conditions. The method developed here leads directly to a new differential equation and boundary conditions for a continuous cracked Euler–Bernoulli beam with a breathing crack.

The method has been tested for the evaluation of the lowest natural frequency of lateral vibration for beams with a single-edge breathing crack. For the analysis, the method for the breathing crack developed here was compared with the continuous cracked beam vibration theory for open crack vibration analysis. Experimental results were used to validate the theory developed.

The frequency ratio ω_1^*/ω_1 has been calculated analytically (a) with the continuous crack model, equation (13), and (b) with the breathing crack model, equation (28). Experimental data for an aluminium beam of length 0·235 m, cross-section width 0·006 m, cross-section height 0·0254 m, $E = 2.06\text{E}11\ \text{N/m}^2$, material density $2800\ \text{kg/m}^3$ and Poisson's ratio 0·35, with a breathing and an open crack at mid-span are in good agreement with the analytical results.

The results from the preceding analysis show that in the absence of sufficient preload, fatigue cracks behave as breathing cracks, resulting in a smaller drop in natural frequencies than an open-crack model predicts. This is an important factor in applications of the method for crack identification. According to the preloading conditions of the structure under investigation, either the open-crack model or the breathing crack model must be identified. It is evident that using an open-crack model assumption to interpret vibration measurements for a fatigue-breathing crack will lead to the incorrect conclusion that the crack severity is smaller than what it really is.

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APPENDIX A: NOMENCLATURE

a	crack depth
A	beam cross-sectional area
b	cross-section width
c	local crack flexibility
c_0	material constant
E	Young's modulus of elasticity
e	$= (\gamma_{XX} + \gamma_{YY} + \gamma_{ZZ})$ volume dilatation
$f(x, z)$	crack disturbance function
F_i	body forces
g_i	surface traction
h	cross-section height
I	cross-sectional area moment of inertia
L	length of beam
L_0	distance from crack tip
M	bending moment
p_i	momentum
$p(x, t)$	velocity field
S_g, S_u	external surfaces
$S(x, t)$	strain function
$T(x, t)$	stress function
T_m	kinetic energy density
u_i	displacement field components
U_T	strain energy due to crack
V	total volume of the solid
$W(\gamma_{ij})$	strain energy density function
$w(x, t)$	lateral displacement function
w_0	lateral displacement of the uncracked beam
w^*	lateral displacement due to crack
α	crack ratio a/h
β	non-dimensional crack location
β_n^*	cracked beam natural frequency parameter
γ_{ij}	strain tensor components
δ_{ij}	Kronecker's delta
v	Poisson's ratio
ρ	material density
σ_{ij}	stress tensor components
$\phi(x)$	mode disturbance function
ω_n	natural frequencies of the uncracked beam
ω_n^*	natural frequencies of the cracked beam